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Universality classes in directed sandpile models

LETTER TO THE EDITOR

Universality classes in directed sandpile models

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Abstract. We perform large-scale numerical simulations of a directed version of the two-state stochastic sandpile model. Numerical results show that this stochastic model defines a new universality class with respect to the Abelian directed sandpile. The physical origin of the different critical behaviour has to be ascribed to the presence of multiple topplings in the stochastic model. These results provide new insight into the long-debated question of universality in Abelian and stochastic sandpiles.

The class of sandpile models, consisting of the original Bak, Tang and Wiesenfeld (BTW) [1] automata and its theme variations, is considered the prototypical example of a special class of driven non-equilibrium systems exhibiting a behaviour dubbed self-organized criticality (SOC). Under an external drive, these systems spontaneously evolve into a stationary state. In the limit of infinitesimal driving the stationary state shows a singular response function associated to an avalanche-like dynamics, indicative of a critical behaviour. Sandpile models have thus attracted a great deal of interest, as plausible candidates to explain the avalanche behaviour empirically observed in a large number of natural phenomena [2].

In recent years, the possibility of understanding the sandpile critical behaviour in analogy with other non-equilibrium critical phenomena such as branching processes [3, 4], interface depinning models [5, 6], and absorbing phase transitions [7, 8] has been pointed out. It is then most important to identify precisely, for sandpiles, the universality classes and upper critical dimensions, which are basic and discriminating features of the critical behaviour. Despite significant numerical efforts, however, these issues remain largely unresolved. For instance, it is still an unanswered problem whether or not the original deterministic BTW sandpile and the stochastic Manna two-state model [9] belong to the same universality class. Theoretical approaches [10–12] support the idea of a single universality class, while numerical simulations provide contradictory results [13–15].

In order to have a deeper understanding of the universality classes puzzle, we turn our attention to *directed sandpile models* [16]. In this case Dhar and Ramaswamy obtained an exact solution for the Abelian directed sandpile (ADS) [16] which can be used as a benchmark to check the numerical simulation analysis. Directed sandpiles thus become an interesting test field to study how critical behaviour is affected by the introduction of stochastic elements. Despite the fact that results obtained for directed models cannot be exported 'tout court' to the isotropic ones, the eventual appearance of different universality classes provides interesting clues on the general problem of universality in sandpiles. This issue has been recently addressed in a particular case by Tadić and Dhar [17], but a general discussion of universality classes in directed sandpile automata is still lacking.
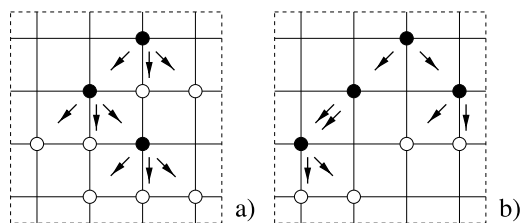
Figure 1. Toppling rules in $d = 2$ for directed sandpiles. Filled circles represent active (toppling) sites; empty circles are stable sites. In the Abelian model (*a*) an active site sends one grain to each one of its three neighbours on the next downwards row. In the stochastic models (*b*) one grain is sent to two randomly chosen downward neighbours.

In this letter we present large-scale numerical simulations of the ADS and the stochastic directed sandpile (SDS) models. First, we study an ADS model for which we recover numerically the results expected from the analytical solution [16]. Then we introduce a stochastic model which is a directed version of the Manna two-state sandpile [9]. In this case, the set of critical exponents defines a different universality class. For both models we provide a very accurate study of finite-size effects and the convergence to the asymptotic behaviour. For small and medium lattice sizes we find scaling anomalies that are similar to those encountered in isotropic models. We also study in detail the geometrical structure of avalanches. The presence of multiple topplings appears to be the fundamental difference between Abelian and stochastic models. Numerical simulations in Euclidean dimension $d > 2$ show that both universality classes have an upper critical dimension $d_c = 3$, where strong logarithmic corrections to scaling are present.

We consider the following definition for an ADS model (see figure 1(*a*)). On each site of a d-dimensional hypercubic lattice of size L, we assign an integer variable z_i, called 'energy'. At each time step, an energy grain is added to a randomly chosen site ($z_i \rightarrow z_i + 1$). When a site acquires an energy greater than or equal to the threshold $z_c = 2d - 1$, it topples. Topplings are directed along a fixed direction $x_\parallel$ (defined usually as 'downwards'): when a site on the hyperplane $x_\parallel$ topples, it sends *deterministically* one energy grain to each nearest and next-nearest neighbour site on the hyperplane $x_\parallel + 1$, for a total of $2d - 1$ grains. This definition differs from the ADS studied by Dhar and Ramaswamy [16] in the orientation of the lattice. Both models, however, share the same universality class, being Abelian, deterministic, and directed.

The stochastic generalization of the above model is depicted in figure 1(*b*): the threshold is now $z_c = 2$, independent of the spatial dimension. When a site at the hyperplane $x_\parallel$ topples, it sends two grains of energy to two sites, randomly chosen among its $2d - 1$ neighbours in the hyperplane $x_\parallel + 1$. The dynamical rule of this model can be defined *exclusive* if the two energy grains are always distributed on different sites. In contrast, a *nonexclusive* dynamics allows the transfer of two energy grains to the same site. We will consider separately the cases of the exclusive stochastic directed sandpile (ESDS) and the nonexclusive stochastic directed sandpile (NESDS). It is worth remarking that stochasticity does not alter the Abelian nature of the model [18]. All three models are locally conservative; no energy grains are lost during a toppling event. Boundary conditions are periodic in the transverse directions and open at the bottom hyperplane $x_\parallel = L$, from which energy can leave the system.

In the critical stationary state, we can define the probability that the addition of a single grain is followed by an avalanche of toppling events. Avalanches are then characterized by the number of topplings s, and the duration t. According to the standard finite-size scaling

(FSS) hypothesis, the probability distributions of these quantities are described by the scaling functions

$$P(s) = s^{-\tau_s} \mathcal{G}(s/s_c) \tag{1}$$

$$P(t) = t^{-\tau_t} \mathcal{F}(t/t_c) \tag{2}$$

where s_c and t_c are the cut-off characteristic size and time, respectively. In the critical state the lattice size L is the only characteristic length present in the system. Approaching the thermodynamic limit ($L \to \infty$), the characteristic avalanche size and time diverge as $s_c \sim L^D$ and $t_c \sim L^z$, respectively. The exponent D defines the fractal dimension of the avalanche cluster and z is the usual dynamic critical exponent. The directed nature of the model introduces a drastic simplification, since it imposes $z = 1$. A general result concerns the average avalanche size $\langle s \rangle$, that also scales linearly with L [16,19,20]: a new injected grain of energy has to travel, on average, a distance of order L before reaching the boundary. In the stationary state, on average, each energy grain input will correspond to an energy grain flowing out of the system. This implies that the average avalanche size corresponds to the number of topplings needed for a grain to reach the boundary; i.e. $\langle s \rangle \sim L$. The same result can be exactly obtained by inspecting the conservation symmetry of the model [21].

For the ADS, the exact analytical solution in $d = 2$ yields the exponents $\tau_s = \frac{4}{3}$, $\tau_t = \frac{3}{2}$ and $D = \frac{3}{2}$ [16]. The upper critical dimension is found to be $d_c = 3$, and it is also possible to find exactly the logarithmic corrections to scaling [16, 22]. The introduction of stochastic ingredients in the toppling dynamics of directed sandpiles has been studied only recently in a model that randomly stores energy on each toppling [17]. This model is strictly related to directed percolation and defines a universality class *per se*. In our case stochasticity affects only the partition of energy during topplings, and there is no analytical insight for the critical behaviour of this model. In order to discriminate between ADS and SDS we perform simulations of both models for sizes ranging from $L = 100$ to $L = 6400$. Statistical distributions are obtained averaging over 10^7 avalanches. Comparison of numerical results on the ADS allows us to check the reliability and degree of convergence with respect to the lattice sizes used.

It is well known from the many numerical papers on sandpiles that an accurate determination of the exponents τ_s and τ_t is a subtle issue. An overall determination within 10% accuracy is a relatively easy task. However, a truly accurate measurement, allowing a precise discrimination of universality classes, is strongly affected by the lower and upper cut-offs in the distribution. Extrapolations and local slope analysis are often very complicated and the relative error bars are not clearly defined. In this respect, it is far better to calculate exponents by methods that contain the system-size dependence explicitly; namely data collapse and moment analysis. Moment analysis was introduced by De Menech *et al* [23] in the context of the two-dimensional BTW, and it has been used extensively on Abelian and stochastic models [15,24]. The q-moment of the avalanche size distribution on a lattice of size L, $\langle s^q \rangle_L = \int s^q P(s) \, \mathrm{d}s$, has the following size dependence:

$$\langle s^q \rangle_L = L^{D(q+1-\tau_s)} \int y^{q-\tau_s} \mathcal{G}(y) \, \mathrm{d}y \sim L^{D(q+1-\tau_s)} \tag{3}$$

where we have used the transformation $y = s/L^D$ in the FSS from equation (1). More generally, $\langle s^q \rangle_L \sim L^{\sigma_s(q)}$, where the exponents $\sigma_s(q)$ can be obtained as the slope of the log–log plot of $\langle s^q \rangle_L$ versus L. Using equation (3), we obtain $\langle s^{q+1} \rangle_L / \langle s^q \rangle_L \sim L^D$ or $\sigma_s(q+1) - \sigma_s(q) = D$, so that the slope of $\sigma_s(q)$ as a function of q is the cut-off exponent; i.e. $D = \partial \sigma_s(q)/\partial q$. This is not true for small q because the integral in equation (3) is dominated by its lower cut-off. In particular, corrections to scaling are important for $q \leqslant \tau_s - 1$. An additional

Table 1. Critical exponents for directed sandpiles in $d = 2$. DR: Dhar and Ramaswamy's exact result; ADS: Abelian model; ESDS, NESDS: stochastic models.

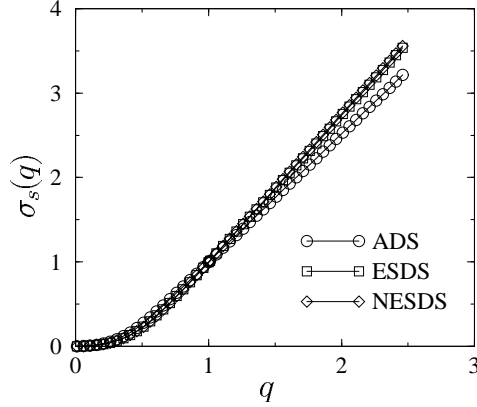| Model	τ_s	D	τ_t	z
DR	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	1
ADS	1.34 ± 0.01	1.51 ± 0.01	1.51 ± 0.02	1.00 ± 0.01
ESDS	1.43 ± 0.01	1.74 ± 0.01	1.71 ± 0.03	0.99 ± 0.01
NESDS	1.43 ± 0.01	1.75 ± 0.01	1.74 ± 0.04	0.99 ± 0.01

Figure 2. Plot of $\sigma_s(q)$ for the $d = 2$ models ADS, ESDS and NESDS.

and strong check on the numerical data can be found in the fact that, as we have previously shown, the first moment of the size distribution must scale linearly with L. This last constraint also allows evaluation of the exponent τ_s from the scaling relation $(2 - \tau_s)D = \sigma_s(1) = 1$, which should be satisfied for large enough sizes.

Along the same lines we can obtain the moments of the avalanche time distribution. In this case $\langle t^q \rangle_L \sim L^{\sigma_t(q)}$, with $\partial \sigma_t(q)/\partial q = z$. Analogous considerations for small q also apply for the time moment analysis. Here, an estimate of the asymptotic convergence of the numerical results is provided by the constraint $z = 1$, that holds for large enough sizes. Then, the τ_t exponent can be found using the scaling relation $(2 - \tau_t) = \sigma_t(1)$.

Despite the fact that the moment method is usually rather accurate, it must be corroborated by a data collapse analysis. The FSS of equations (1) and (2) has to be verified and must be consistent with the numerical exponents obtained from the moment analysis. This can be done by rescaling $s \rightarrow s/L^D$ and $P(s) \rightarrow P(s)L^{D\tau_s}$ and correspondingly $t \rightarrow t/L^z$ and $P(t) \rightarrow P(t)L^{z\tau_t}$. Data for different L must then collapse onto the same universal curve if the FSS hypothesis is satisfied. Complete consistency between the methods gives the best collapse with the exponents obtained by the moments analysis. In table 1 we report the exponents found for the ADS, ESDS and NESDS in $d = 2$. Figure 2 shows the moments $\sigma_s(q)$. Figures 3 and 4 plot the FSS data collapse for sizes and times, respectively.

The exponents obtained for the ADS are in perfect agreement with the expected analytical results. This fact supports the idea that the system sizes used in the present work allow one to recover the correct asymptotic behaviour. It is worth remarking that, for small and medium lattice sizes, both moments and data collapse analysis present scaling features that cannot be reconciled in the single scaling picture usually considered. These anomalies are not persistent and disappear for reasonably large sizes ($L \simeq 10^3$). This evidence for a slow decaying of
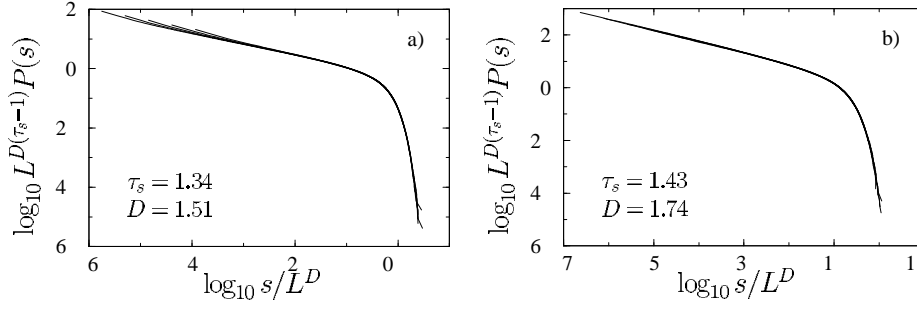
Figure 3. Data collapse analysis of the integrated avalanche size distribution for the $d = 2$ models: (*a*) ADS and (*b*) ESDS. System sizes are $L = 400, 800, 1600, 3200$ and 6400. The results for the ESDS and NESDS are identical, within error bars.
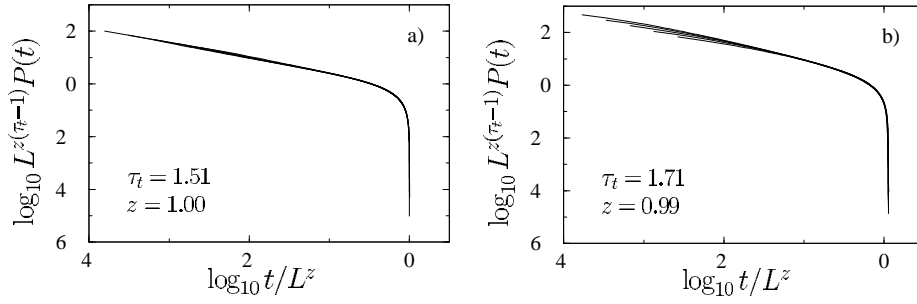

Figure 4. Data collapse analysis of the integrated avalanche time distribution for the $d = 2$ models (*a*) ADS and (*b*) ESDS. System sizes are $L = 400, 800, 1600, 3200$ and 6400. The results for the ESDS and NESDS are identical, within error bars.
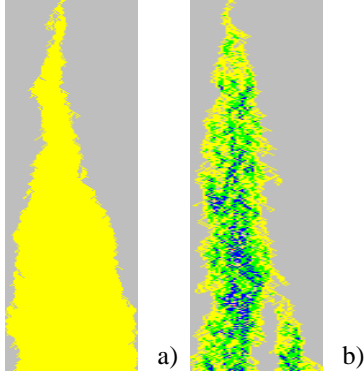

Figure 5. Colour plots of the local density of topplings in two avalanches of size 50 000, in $d = 2$, for (*a*) ADS and (*b*) ESDS. White represents a single toppling per site; black stands for the maximum number of topplings.

finite-size effects could shed light on several anomalies reported in isotropic sandpiles, for which, unfortunately, it is very difficult to reach very large sizes [15, 23, 24]. Results for the ESDS and NESDS are identical within the error bars, indicating that these two models are in the same universality class. On the other hand, the obtained exponents show, beyond any doubt, that Abelian and SDS models do not belong to the same universality class.

The compelling numerical evidence for two distinct universality classes does not tell us what is the basic mechanism at the origin of the different critical behaviour. In order to have a deeper insight into the dynamics of the various models, we have inspected the geometric
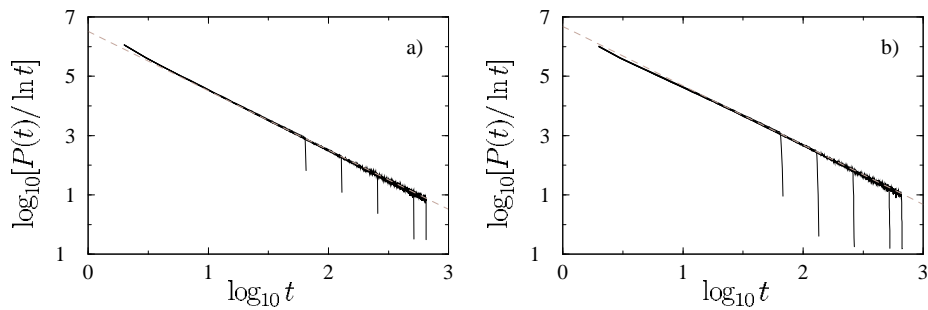
Figure 6. Data collapse with logarithmic corrections of the avalanche time distribution for the $d = 3$ models (*a*) ADS and (*b*) ESDS. The dashed lines have slope -2. System sizes are $L = 64$, 128, 256, 512 and 650.

structure of the resulting avalanches. In figure 5 we depict in a colour plot the local density of topplings in two avalanches of size 50 000 corresponding to the two-dimensional ADS and ESDS models. From the figure it becomes apparent that the stochastic dynamics introduces multiple toppling events, which are by definition absent in the Abelian case. This gives rise to very different avalanche structures, eventually reflected in the asymptotic critical behaviour. In particular, the fractal dimension D is indicative of the scaling of toppling events with sizes. In the stochastic case we recover a higher fractal dimension than in the Abelian case. The multiple toppling mechanism has been proposed in the past as the origin of differences between *isotropic* Abelian and stochastic sandpiles as well. In that case, however, multiple toppling is a common feature of both models, and for the largest sizes reached so far, they share the same fractal dimension D [15].

Analysis of the models in three dimensions is strongly hindered by the presence of logarithmic corrections [16, 22]. Nonetheless, a naive application of the moment analysis yields values compatible with the mean-field results $\tau_s = \frac{3}{2}$, $\tau_t = 2$ and $D = 2$ [16]. More interestingly, in [16] the authors were able to deduce the exact form of the logarithmic corrections in $d = 3$ for the avalanche time distribution, namely $P(t) \sim t^{-2} \ln t$. In figure 6 we have checked that the same logarithmic corrections apply to both the Abelian and ESDS sandpiles. This remarkable fact lends support to the critical dimension of the stochastic model being $d_c = 3$.

In summary, we have reported large-scale numerical simulations of a SDS model. This model defines unambiguously a different universality class with respect to the ADS model. The origin of this difference is traced back to the avalanche cluster geometric structure, providing new clues to understand the effect of stochastic elements in the dynamics of avalanche processes.

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